Theory for the Estimation of Structural Vibration Parameters from Incomplete Data

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Introduction

IN system identification and modal analysis, the investigator is often faced with the problem of having to construct a model of finite order to represent a continuous vibrating structure. In other words, he is charged with obtaining a reduced-order model which, by definition, must be limited in its representation of the continuous system to a finite range of frequencies. The quality of the reduced-order model will be contingent upon the completeness of the measured data: a common cause of incompleteness occurs when the number of modes excited in the test is fewer than the number of measurement stations (i.e., less than the order of the identified model).

When it is required to construct a structural model (in terms of mass, stiffness, and damping parameters) from measured data, it is vital that such a model should in some sense be physically meaningful. The problems associated with the construction of reduced-order structural models from incomplete data have been highlighted by Berman and Flannelly[1] and Berman.[2] It is obvious that a reduced-order model cannot possibly reproduce the behavior of a continuous structure across an infinite range of frequencies. In the real structure the mass is distributed, whereas in the model the mass is discretized. In the finite-element method, consistent mass and stiffness matrices may be formulated by the minimization of an energy functional and, for this reason, such matrices may be considered to possess physical meaning. In reality, finite-element models do not exactly represent the dynamics of vibrating structures because they are incapable of dealing with boundary conditions such as imperfect hinges or flexibility in welded joints. It is then necessary to modify the finite-element model slightly in order that it should replicate the observed dynamics of the vibrating structure in a required range of frequencies.

Methods have been developed for the estimation of structural parameters based on measured modal data. Chen and Garba[3] presented a least-squares technique that may be applied iteratively together with matrix perturbation (for computation of the Jacobian matrix and reanalysis of the eigenvalues and eigenvectors) until the measured and computed eigendata are sufficiently in agreement. However, modal methods are known to have difficulty in treating closely spaced modes and can only deal with out-of-range modes by the addition of residual flexibilities.

In this Note, a method is presented that does not depend on a modal decomposition but uses measured frequency responses to construct a structural model that varies minimally from an initial finite-element representation. A model constructed by this method is thus physically meaningful in the sense that a finite-element model is meaningful. A simple problem is contrived to illustrate an extreme case of incomplete data, which is solved using a continuous-frequency-domain, least-squares filter.[4] Using this approach, the Jacobian matrix is a constant matrix. Other problems associated with measurement noise are not discussed here since the filtering method has already been shown to perform well using experimental data from a portal frame rig.[5]

Theory

Consider an m-degree-of-freedom vibrating system that may be described in a finite range of frequencies by the following model

$$q(\omega) = B(\omega)[z(\omega) - \xi] \tag{1}$$

where $q(\omega)$ is an n-vector of input forces, $z(\omega)$ an n-vector of measured displacement responses, ξ an n-vector of measurement noise, $B^{-1}(\omega)$ an $n \times n$ matrix of frequency response functions, and $n < m$.

In the case of viscous damping,

$$B(\omega) = -\omega^2 M + j\omega C + K \tag{2}$$

where M, C, and K are $n \times n$ matrices of mass, viscous damping, and stiffness, respectively.

Equation (1) may be arranged as follows:

$$H^H x = q - D^H y \tag{3}$$

where x is an ℓ-vector of *unknown* mass, damping, and stiffness parameters and y is a $(k\text{-}\ell)$-vector of *known* mass, damping, and stiffness parameters. For a symmetric system, $k = 3\,(n^2 + n)/2$.

$$H = \frac{\mathrm{d}(Bz)^H}{\mathrm{d}x} \tag{4}$$

$$D = \frac{\mathrm{d}(Bz)^H}{\mathrm{d}y} \tag{5}$$

where H is the Jacobian matrix. Clearly Eq. (3) allows for imperfect measurement of the vibration responses.

If $r_i = q_i - D_i^H y$, then a least-squares estimate $\hat{x}$ of x can be obtained by solving the following:

$$\sum_{i=1}^{N} H_i^H \hat{x} = \sum_{i=1}^{N} r_i \tag{6}$$

where terms on either side with the same summation index, i, are also equal and where frequency-domain data is sampled at N intervals.

When an initial (finite-element) model x_{fe} is available then subtracting

$$\sum_{i=1}^{N} H_i^H x_{\mathrm{fe}}$$

from both sides of Eq. (6) and multiplying throughout by $H_i Q$

$$\sum_{i=1}^{N} H_i Q H_i^H (\hat{x} - x_{\mathrm{fe}}) = \sum_{i=1}^{N} H_i Q (r_i - H_i^H x_{\mathrm{fe}}) \tag{7}$$

where Q is a positive definite weighting matrix.

If $e = (\hat{x} - x_{\mathrm{fe}})$ represents a least-squares correcting term that must be added to the finite-element model, then

$$e = \left(\sum_{i=1}^{N} H_i Q H_i^H \right)^{+} \sum_{i=1}^{N} H_i Q (r_i - H_i^H x_{\mathrm{fe}}) \tag{8}$$

If

$$\mathrm{rank}\left(\sum_{i=1}^{N} H_i Q H_i^H \right) = \ell$$

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then $(\ )^+ = (\ )^{-1}$, and a unique solution is available for $\boldsymbol{e}$. It is more likely that

$$\left(\sum_{i=1}^{N} H_i Q H_i^H\right)$$

is rank deficient, in which case there will be a multiplicity of least-squares solutions and then the pseudoinverse should be computed using singular value decomposition, which results in the unique solution of smallest norm $\|\boldsymbol{e}\|_2$. Thus the singular value decomposition forces the solution $\hat{\boldsymbol{x}}$, which deviates least from the initial model $\boldsymbol{x}_{fe}$. The reader is referred to Ref. 6, pp. 424–325, for a discussion of the application of singular value decomposition to degenerate least-squares problems.

The result [Eq. (8)] may be posed as a discrete, frequency domain filter. Consider the case of a single frequency step from $i = j - 1$ to $i = j$; then the estimate $\hat{x}_j$ can be computed from $\hat{x}_{j-1}$ as follows:

$$\hat{x}_j - \hat{x}_{j-1} = \left(\sum_{i=1}^{N} H_i Q H_i^H\right)^+ H_j Q (r_j - H_j^H \hat{x}_{j-1}) \tag{9}$$

or

$$\Delta e_j = \left(\sum_{i=1}^{N} H_i Q H_i^H\right)^+ H_j Q [r_j - H_j^H (x_{fe} + e_{j-1})] \tag{10}$$

where

$$e_{j-1} = (\hat{x}_{j-1} - x_{fe}) \tag{11}$$

$$e_j = e_{j-1} + \Delta e_j \tag{12}$$

If the frequency interval $\Delta\Omega = \Omega_j - \Omega_{j-1}$, then

$$\frac{\Delta e_j}{\Delta\Omega} = \left(\sum_{i=1}^{N} H_i Q H_i^H \Delta\Omega\right)^+ H_j Q [r_j - H_j^H (x_{fe} + e_{j-1})] \tag{13}$$

and in the limit as $\Delta\Omega \to 0$,

$$\frac{de}{d\Omega} = (P^{-1})^+ HQ[r - H^H(x_{fe} + e)] \tag{14}$$

where

$$\frac{dP^{-1}}{d\Omega} = HQH^H \tag{15}$$

Equations (14) and (15) together describe a continuous, frequency-domain filter for the estimation of the correcting term $\boldsymbol{e}$.

Simulated Experiment

A simulated experiment was conducted using a finite-element representation of a simply supported beam. The beam of length $\ell = 32$, mass per unit length $m = 0.1$, and rigidity $EI = 1.47 \times 10^6$ was modeled using eight beam elements. The

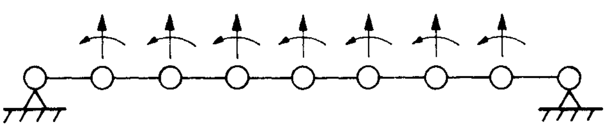

Fig. 1a Finite-element models, 14-degree-of-freedom model.

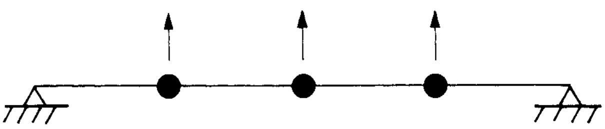

Fig. 1b Finite-element model, reduced 3-degree-of-freedom model.

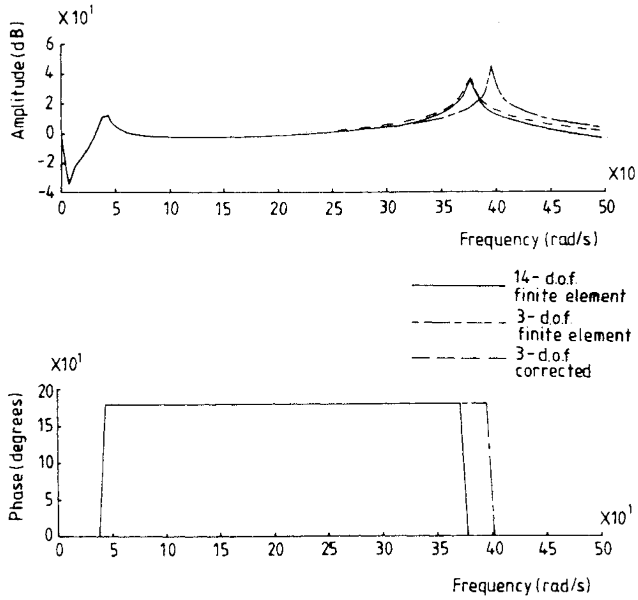

Fig. 2a Sample acceleration spectra, loading at midspan.

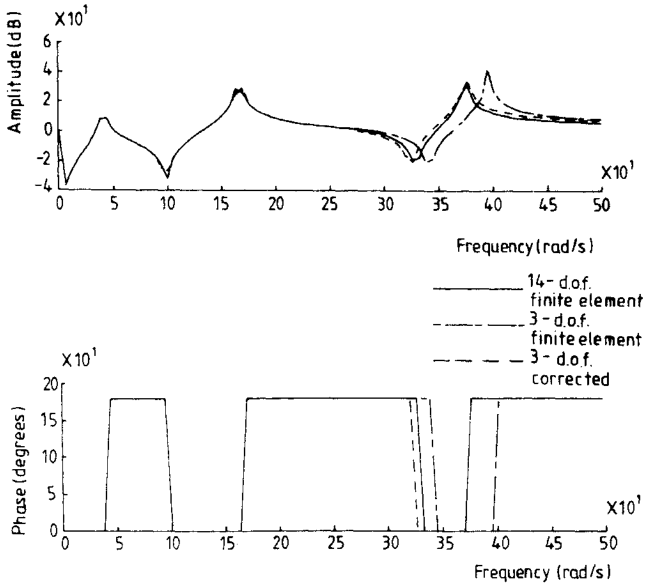

Fig. 2b Sample acceleration spectra, loading at one-quarter-span.

finite-element model contained 14 degrees-of-freedom (DOF), but it was required to set up a 3-degree-of-freedom model as shown in Fig. 1. Unit white noise excitation was applied in the transverse direction at the center of the beam such that the even modes were not excited. A 3-DOF finite-element model was set up using Guyan reduction, and the lower triangles of the mass and stiffness matrices were entered into x_{fe}. The correcting term $\boldsymbol{e}$ was then computed using displacement data that were generated by the 14-DOF model but recorded only at the three measurement stations of the reduced model. Finally the two finite-element models and the least-squares model $\hat{x}$ were compared when the applied load was removed from the center of the beam and reapplied at one-quarter span. In this latter case, the first three modes were all excited.

In the processing of the measured data, two small singular values were set to zero resulting in a reduction of the condition number from 10^{16} to around 200. A typical acceleration spectrum for the two finite-element models and the corrected 3-DOF model in the range 0–500 rad/s with loading at midspan is shown in Fig. 2a. The natural frequencies (which were obtained from the eigenvalues) of the same three models

Table 1 Computed natural frequencies

Finite element			Corrected	
14-DOF	3-DOF		3-DOF	
rad/s	rad/s	% error	rad/s	% error
41.74	41.75	0.03	41.74	0.0008
166.99	168.15	0.69	174.77	4.6600
376.16	397.62	5.71	376.15	−0.0030

Table 2 Estimated parameters

	Reduced finite-element model	Correcting term	Corrected model
	$\times 10^5$		
k_{11}	0.28351	−0.01343	0.27008
k_{21}	−0.27119	0.02859	−0.24259
k_{22}	0.39445	−0.04031	0.35414
k_{31}	0.11094	−0.02686	0.08408
k_{32}	−0.27119	0.02859	−0.24259
k_{33}	0.28351	−0.01343	0.27008
	$\times 10^{-1}$		
m_{11}	5.8509	0.01424	5.8651
m_{21}	0.47902	0.02570	0.50472
m_{22}	5.5981	0.04427	5.6423
m_{31}	−0.25279	0.02848	−0.22432
m_{32}	0.47902	0.02570	0.50472
m_{33}	5.8509	0.01424	5.8651

are given in Table 1, and reduced finite-element and corrected parameters are provided in Table 2. It should be noted that the corrected model retains the natural symmetry of the simply supported beam problem (i.e., $k_{11} = k_{33}$, $k_{21} = k_{32}$, $m_{11} = m_{33}$, and $m_{21} = m_{32}$). Figure 2b shows an acceleration spectrum obtained after the load had been moved to one-quarter span.

The corrected model is shown to be significantly more accurate than a Guyan-reduced model in predicting the first and third eigenvalues. In the computation of the second eigenvalue, the reduced model is marginally better than the corrected model. Overall, the corrected model is superior to the reduced model in replicating the measured acceleration spectra. The estimated parameters deviate minimally from the parameters of the initial model and may be considered to be physically meaningful in the sense that a finite-element model is meaningful. As a final test of goodness, the model is able to predict accurately the response of the beam to loading conditions which differ from the loads applied in the test.

Acknowledgment

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Effects of Curvature on Composite Material Beams

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Nomenclature

A'	= laminate A' matrix
a	= constant
b	= flange width
c_1,c_2,c_3,c_4	= constants
D	= laminate D matrix
f	= flange
$FBSC$	= flange bending stiffness coefficient
k_1,k_2,k_3,k_4,k_5	= constants
LC	= lip coefficient
l	= length, l-subscript: lip
M	= laminate moment resultant
N	= laminate stress resultant
NA	= neutral axis
Q	= laminate transverse shear stress resultant
q	= transverse shear loading
R	= radius to flange midplane
$RRSC$	= root radial stiffness coefficient
RS	= radial support
$RTBSC$	= root transverse bending stiffness coefficient
t	= thickness
u	= superscript: undeformed
u,v,w	= displacement in circumferential, y, z or, r directions, respectively
w	= subscript: web
$WPRC$	= web Poisson's ratio coefficient
α	= effectivity
β	= transverse bending factor
ϵ	= strain
σ	= stress

Much of the nomenclature is based on classical lamination theory.[1]

Introduction

CURVATURE can have a significant effect on the stress distributions and failure modes of laminated composite material beams as compared to predicted behavior using isotropic material properties. This Note presents a classical analysis procedure for curved composite material beams with thin flanges.

Curvature effects on beams with wide flanges are illustrated in Fig 1. The major effects of an applied moment in the plane of beam curvature can be summarized as follows.

1) The neutral axis of the beam section shifts toward the concave side.

2) The axial bending stress distribution is nonlinear in the radial direction.

3) The flange axial load induces radial flange loading.

4) The induced radial flange loading causes a transverse flange bending moment and a nonlinear axial bending stress distribution in the flange width direction.

5) The transverse flange bending moment induces interlaminar (radial) stresses in the curved flange root area.

These phenomena (items 1 to 4) have been studied extensively for isotropic materials.[2-4] Design curves are readily

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